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On a Classification of Prehomogeneous Vector Spaces over Local and Global Fields

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INTRODUCTION

Let G be a connected reductive algebraic group over a field F and let ρ be a rational representation of G on a finite dimensional vector space V defined over F. A triplet (G, ρ, V) is called a prehomogeneous vector space if G has a Zariski open orbit in V over the algebraic closure $\overline{F}$ of F. In prehomogeneous vector spaces, there is an important class called reduced irreducible regular. In a fundamental paper, Sato and Kimura [S-K] gave a classification of reduced irreducible regular prehomogeneous vector spaces over C. In [R], Rubenthaler classified reduced irreducible regular prehomogeneous vector spaces of parabolic type over R. However, to study their arithmetic properties, especially their zeta functions, it is necessary to know their forms over algebraic number fields. The purpose of this paper is to classify forms of reduced irreducible prehomogeneous vector spaces over non-archimedean local fields of characteristic 0 and algebraic number fields.

More precisely, we explain what we will actually classify. Two prehomogeneous vector spaces (G, ρ, V) and (G', ρ', V') are called equivalent to each other if there exist isomorphisms

$$\sigma : \rho(G) \to \rho'(G'), \qquad \tau : V \to V'$$

such that

$$\tau(gv) = \sigma(g)\tau(v), \qquad g \in \rho(G), v \in V.$$

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We say they are equivalent to each other over F if there exist σ and τ as previously defined over F. For (g, ρ, V), let $G = TG_{\mathrm{der}}$ with the central torus T and the derived group G_{der} of G. Let $\tilde{G}_{\mathrm{der}}$ be the universal covering group of G_{der} and $\pi\colon \tilde{G}_{\mathrm{der}} \to G_{\mathrm{der}}$ be the natural projection. Here we call the triplet $(\tilde{G}_{\mathrm{der}}, \rho \circ \pi, V)$ the semisimple part of (G, ρ, V). Set $\tilde{G} = T \times \tilde{G}_{\mathrm{der}}$ and $\tilde{\rho}((a, g)) = \rho(a)\rho(\pi(g))$. Then $(\tilde{G}, \tilde{\rho}, V)$ defines a prehomogeneous vector space and we can recover (G, ρ, V) from $(\tilde{G}, \tilde{\rho}, V)$ easily.

Let us consider triplets (H, ρ, V) consisting of a connected semisimple simply connected algebraic group H and its representation ρ on a vector space V. If H and ρ are defined over F, we say (H, ρ, V) are defined over F. Let (H', ρ', V') be another such triplet. We call (H, ρ, V) equivalent to (H', ρ', V') if there exist isomorphisms

$$f\colon H \to H', \qquad l\colon V \to V'$$

such that

$$l(\rho(g)v) = \rho'(f(g))l(v), \qquad g \in H, v \in V.$$

If there exist f and l defined over F, we say they are equivalent to each other over F. Then we easily see two prehomogeneous vector spaces are equivalent if and only if their semisimple parts are equivalent. In this paper, we will give a classification of $(\tilde{G}_{\mathrm{der}}, \tilde{\rho}, V)$ over F for F a non-archimedean local field or an algebraic number field by using Galois cohomology for reduced irreducible (G, ρ, V).

On the torus part, we recall the following. Let T' be a torus and let λ' be a nontrivial character of T' into $\mathbf{G}_m$ defined all over F. Set $\tilde{G}' = T' \times \tilde{G}_{\mathrm{der}}$, $\tilde{\rho}'((a, g)) = \lambda'(a)\rho(\pi(g))$, and $(a, g) \in T \times \tilde{G}_{\mathrm{der}}$. Then $(\tilde{G}', \tilde{\rho}', V)$ also defines a prehomogeneous vector space over F for any such pair (T', λ'), and we obtain various $(\tilde{G}', \tilde{\rho}', V)$ by changing (T', λ'). To study zeta functions of prehomogeneous vector spaces, it seems natural to assume $\operatorname{Ker} \rho = \{1\}$ by taking the quotient if necessary. Then the choice of (T', λ') is irrelevant and we may assume $G = H \times \mathbf{G}_m / \operatorname{Ker} \rho \otimes 1$ for a triplet (H, ρ, V) with a connected semisimple simply connected H as before, where $(\rho \otimes 1)((g, a)) = a\rho(g)$. For such G we can prove the Hasse principle for $H^1(F, G)$, which is useful in the study of zeta functions.

In Section 1, we describe the equivalence classes of (G, ρ, V) semisimple parts of reduced irreducible prehomogeneous vector spaces by means of Galois cohomology. Our idea is based on [B-T] and [T1]. In Section 2, we determine explicitly the equivalence classes for irreducible reduced prehomogeneous vector spaces for F a non-archimedean local field or an algebraic number field by calculating Galois cohomology. In the case of algebraic number fields, we reduce the calculation to that on local fields by the Hasse principle. The calculation over local fields will be done completely for non-archimedean fields. On R, the classification was completed by Rubenthaler in the case of parabolic type, and in the other cases the Galois cohomology is known by classical results.

1. PREHOMOGENEOUS VECTOR SPACES

Let F be a field of characteristic 0, $\overline{F}$ its algebraic closure, and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. A prehomogeneous vector space (G, ρ, V) is called irreducible if ρ is irreducible over $\overline{F}$, and is called reduced if it has the minimum dimension under castling transformations.

In the following text, we consider triplets (G, ρ, V), where G is a connected semisimple simply connected algebraic group defined over F and ρ is a representation of G on a finite dimensional vector space V defined over F. We assume $(G \times \mathbf{G}_m,\ \rho \otimes 1,\ V)$ is an irreducible prehomogeneous vector space, where $(\rho \otimes 1)((g, a))v = a\rho(g)v$. Hence (G, ρ, V) is the semisimple part of a prehomogeneous vector space. Two such triplets, (G, ρ, V) and (G', ρ', V'), are called equivalent over F if there exist isomorphisms f: $G \simeq G'$ and l: $V \simeq V'$ defined over F which satisfy

$$l(\rho(g)v) = \rho'(f(g))l(v), \qquad g \in G, v \in V.$$

We classify F-forms of them.

For G as before, let G_0 be the split group over F which is isomorphic to G over $\overline{F}$, let T_0 be a split torus of G_0, and let B_0 be a Borel subgroup of G_0 containing T_0 defined over F. Let Δ_0 be the set of simple roots of G_0 for the pair (B_0, T_0). Let $\mathrm{Aut}(G_0)$ be the group of automorphisms of G_0. Then there exists a finite subgroup Θ_0 of $\mathrm{Aut}(G_0)$ such that each $\theta \in \Theta_0$ preserves T_0, B_0, and Δ_0, is defined over F, and $\mathrm{Aut}(G_0)$ is a semidirect product of Θ_0 and $\mathrm{Int}(G_0)$, the subgroup of inner automorphisms, which is isomorphic to $\overline{G}_0 = G_0/Z(G_0)$ for the center $Z(G_0)$ of G_0. For these groups, we have a split exact sequence

$$1 \to \mathrm{Int}(G_0) \to \mathrm{Aut}(G_0) \to \Theta_0 \to 1.$$

It is known that the Galois cohomology $H^1(F, \mathrm{Aut}(G_0))$ classifies algebraic groups over F which are isomorphic to G_0 over $\overline{F}$ and $H^1(F, \Theta_0)$ classifies quasisplit groups (cf. [S]). For a class a in $H^1(F, \Theta_0)$, let f_a: $G_0 \to {}_aG_0$ be an isomorphism of G_0 onto a quasisplit algebraic group ${}_aG_0$, defined over $\overline{F}$, which gives a 1-cocycle in the class a. Then ${}_aB_0 = f_a(B_0)$ and ${}_aT_0 = f_a(T_0)$ are a Borel subgroup and a maximal torus of ${}_aG_0$ defined over F. Let ${}_a\Delta_0$ be the set of simple roots for $({}_aB_0, {}_aT_0)$.

Assume G is an inner form of ${}_aG_0$ for $a \in H^1(F, \Theta_0)$. Then there exists an isomorphism $f\colon {}_aG_0 \to G$ such that $f^{-1\sigma}f \in \mathrm{Int}({}_aG_0)$ for $\sigma \in \Gamma$ and the 1-cocycle $(f^{-1\sigma}f)$ defines a class c in $H^1(F, \mathrm{Int}({}_aG_0)) \simeq H^1(F, {}_a\overline{G}_0)$ for ${}_a\overline{G}_0 = {}_aG_0/Z({}_aG_0)$. Let ${}_a\Theta_0 = f_a \circ \Theta_0 \circ f_a^{-1}$ and

$${}_a\Theta_0^\Gamma = \left\{\theta \in {}_a\Theta_0 | {}^\sigma\theta = \theta \;\; \forall \sigma \in \Gamma\right\}.$$

Then ${}_a\Theta_0^\Gamma$ acts on $H^1(F, {}_a\overline{\Theta}_0)$ as follows. Let $f\colon {}_aG_0 \to G$ be as before. Then $(f \circ \theta)^{-1}(f \circ \theta) \in \mathrm{Int}({}_aG_0)$, and the class of the 1-cocycle defined by $f \circ \theta\colon {}_aG_0 \to G$ gives the class $\theta(c)$, and $H^1(F, \mathrm{Int}({}_a\overline{G}_0))/{}_a\Theta_0^\Gamma$ classifies F-forms of G which are inner forms of ${}_aG_0$.

From each isomorphism class over F of inner forms of ${}_aG_0$, we choose G and fix f as before. Let $T = f({}_aT_0)$ and $B = f({}_aB_0)$. We can choose f so that T is defined over F. Let Δ be the set of simple roots of G with respect to T, B. Then f induces a Γ isomorphism between Δ and ${}_a\Delta_0$ (cf. [B-T], Sect. 6). Here the action of Γ on ${}_a\Delta_0$ is the natural one.

Let ρ be a representation of G defined over F on V as previously, and let l_ρ be its highest weight. Then it is known that l_ρ is invariant under Γ. Let ${}_al_\rho = l_\rho \circ f$ be the weight of ${}_aG_0$ to corresponding to l_ρ by f. Since the action of Γ on ${}_a\Delta_0$ is the natural one, we have a representation ${}_a\rho$ of ${}_aG_0$ on a vector space W over F with the highest weight ${}_al_\rho$ (cf. [T1]). Comparing the highest weights, we see ${}_a\rho \circ f^{-1}$ and ρ are equivalent to each other over $\overline{F}$, that is, ρ can be obtained as a F-form of ${}_a\rho \circ f^{-1}$. It is known that ρ is determined by ${}_a\rho \circ f^{-1}$ up to F-equivalence. Conversely, let $({}_aG_0, {}_a\rho, W)$ be a triplet as previously defined over F such that $({}_aG_0 \times \mathbf{G}_m, {}_a\rho \otimes 1, W)$ is an irreducible prehomogeneous vector space, and let h be an isomorphism of ${}_aG_0$ to an algebraic group G defined over $\overline{F}$ such that $h^{-1\sigma}h \in \mathrm{Int}({}_aG_0)$ for $\sigma \in \Gamma$. If ${}_a\rho \circ h^{-1}$ is equivalent to a representation ρ on V defined over F, then we obtained the semisimple part (G, ρ, V) of a prehomogeneous vector space.

Let $h\colon {}_aG_0 \to G$ be an isomorphism as before. We seek the condition under which ${}_a\rho \circ h^{-1}$ is equivalent to a representation rational over F. For this purpose, we consider a map from $H^1(F, {}_a\overline{G}_0)$ to $H^1(F, \mathrm{PGL}(W))$ induced by ${}_a\rho$. Let $a = (\bar{g}_\sigma)$ be a 1-cocycle in ${}_a\overline{G}_0$. Choose $g_\sigma \in {}_aG_0$ so that the class of g_σ in ${}_a\overline{G}_0$ is in $\bar{g}_\sigma$ and the map $\sigma \to g_\sigma$ is continuous. Then $g_{\sigma\tau}(g_\sigma{}^\sigma g_\tau)^{-1}$ is contained in $Z({}_aG_0)$. Let $h_\sigma = {}_a\rho(g_\sigma)$. Since ${}_a\rho$ is irreducible, ${}_a\rho(Z({}_aG_0))$ is contained in the center of $\mathrm{GL}(W)$. Let $\bar{h}_\sigma =$ the class of h_σ in $\mathrm{PGL}(W)$. Then $(\bar{h}_\sigma)$ defines a 1-cocycle in $\mathrm{PGL}(W)$, and this induces a map ${}_a\rho_*$ of $H^1(F, {}_a\overline{G}_0)$ to $H^1(F, \mathrm{PGL}(W))$.

PROPOSITION 1.1. *Let ${}_aG_0$ be a quasisplit semisimple simply connected group over F and ${}_a\rho$ a representation of ${}_aG_0$ on W defined over F. Let G be* *an algebraic group over F, and h a isomorphism of ${}_aG_0$ to G such that $h^{-1\sigma}h = \mathrm{Int}_{\bar{g}_\sigma}$ with $\bar{g}_\sigma \in {}_a\overline{G}_0$ for $\sigma \in \Gamma$. Let c be the class in $H^1(F, {}_a\overline{G}_0)$ defined by $(\bar{g}_\sigma)$. Then the representation ${}_a\rho \circ h^{-1}$ of G is equivalent to a representation rational over F if and only if ${}_a\rho_*(c)$ is trivial in $H^1(F, \mathrm{PGL}(W))$.*

Proof. For each $\bar{g}_\sigma \in {}_a\overline{G}_0$, choose $g_\sigma \in {}_aG_0$ as above. If ${}_a\rho \circ h^{-1}$ is equivalent to a representation over F, then there exists $A \in \mathrm{GL}(W)$ such that

$$ {}^\sigma\!\left(A\left({}_a\rho \circ h^{-1}\right)A^{-1}\right) = A\left({}_a\rho \circ h^{-1}\right)A^{-1}, $$

for all $\sigma \in \Gamma$. Since ${}_a\rho$ is defined over F, we obtain

$$ \begin{aligned} {}_a\rho &= {}^\sigma\!A^{-1}A\left({}_a\rho \circ h^{-1} \circ {}^\sigma h\right)A^{-1\sigma}A \\ &= {}^\sigma\!A^{-1}A\left({}_a\rho \circ \left(\mathrm{Int}_{\bar{g}_\sigma}\right)\right)A^{-1\sigma}A \\ &= \left({}^\sigma\!A^{-1}A\,{}_a\rho(g_\sigma)\right){}_a\rho\left({}^\sigma\!A^{-1}A\,{}_a\rho(g_\sigma)\right)^{-1}. \end{aligned} $$

Hence ${}^\sigma\!A^{-1}A\,{}_a\rho(g_\sigma)$ is contained in the center of $\mathrm{GL}(W)$, and ${}_a\rho_*(c)$ is the class of the 1-cocycle $(\overline{A}^{-1\sigma}\overline{A})$, hence trivial in $H^1(F, \mathrm{PGL}(W))$, where $\overline{A}$ is the class of A in $\mathrm{PGL}(W)$.

Conversely, assume ${}_a\rho_*(c)$ is trivial. Then we can find $A \in \mathrm{GL}(W)$ such that ${}_a\rho(g_\sigma) \equiv A^{-1\sigma}A$ in $\mathrm{PGL}(W)$, and we can trace the preceding argument in reverse. This completes the proof.

Let (G_0, ρ_0, V_0) be a triplet as before with G_0 a split group defined over F. Let l_0 be the highest weight of ρ_0. Assume a triple (G, ρ, V) defined over F is equivalent to (G_0, ρ_0, V_0) over $\overline{F}$. Then there exists an isomorphism $h\colon G_0 \to G$ such that the representations $\rho_0 \circ h^{-1}$ and ρ are equivalent to each other. Assume G is an inner form of ${}_aG_0$ for $a \in H^1(F, \Theta_0)$ and let f and c be as before. Changing h by an inner automorphism, we may assume $h = f \circ f_a \circ \theta$ for $\theta \in \Theta_0$. The highest weight for $\rho \circ f$ is $l_0 \circ \theta^{-1} \circ f_a^{-1}$ and is invariant under Γ. If there exists such (G, ρ, V), taking $f_a \circ \theta$ instead of f_a, we may assume ${}_al_0 = l_0 \circ f_a^{-1}$ is invariant under Γ. Let ${}_a\rho_0$ be the representation of ${}_aG_0$ with the highest weight ${}_al_0$ defined over F. Here we assume the following condition on ${}_al_0$:

$$ \left({}_al_0 \circ {}_a\Theta_0\right)^\Gamma = {}_al_0 \circ {}_a\Theta_0^\Gamma. \tag{1.1} $$

This condition can be checked easily for reduced irreducible prehomogeneous vector spaces. For example, we can show easily this condition is satisfied if

$$ \left\{\theta \in {}_a\Theta_0 \mid {}_al_0 \circ \theta = {}_al_0\right\} = \{1\}. $$

Under (1.1), if $l_0 \circ \theta^{-1} \circ f_a^{-1} = {}_a l_0 \circ (f_a \circ \theta^{-1} \circ f_a^{-1})$ is invariant under Γ, then $l_0 \circ \theta^{-1} \circ f_a^{-1} = {}_a l_0 \circ \theta'^{-1}$ for $\theta' \in {}_a\Theta_0^{\Gamma}$. Hence $\rho \circ f$ is equivalent to ${}_a\rho_0 \circ \theta'^{-1}$ and ρ is equivalent to ${}_a\rho_0 \circ \theta'^{-1} \circ f^{-1} = {}_a\rho_0 \circ (f \circ \theta')^{-1}$. When ${}_a\rho_0 \circ (f \circ \theta')^{-1}$ is equivalent to a representation on $V_{\theta'}$ defined over F, we denote it by $\rho_{\theta'}$. Let

$$ {}_a\Theta_0^{\Gamma, \rho_0} = \left\{ \theta \in {}_a\Theta_0^{\Gamma} \,|\, {}_a l_0 \circ \theta = {}_a l_0 \right\}. $$

Then, Proposition 1.1 and the foregoing consideration prove half of the following theorem.

THEOREM 1.2. *Let* (G_0, ρ_0, V_0) *and* l_0 *be as before. Assume* ${}_a l_0$ *is invariant under* Γ *for* $a \in H^1(F, \Theta_0)$ *and the condition* (1.1) *is satisfied. Then the equivalence classes of triplets* (G, ρ, W) *over F with G an inner form of* ${}_aG_0$, *which are equivalent to* (G_0, ρ_0, V_0) *over* $\overline{F}$, *are in one-to-one correspondence with*

$$ \left(\operatorname{Ker} {}_a\rho_*:\, H^1\left(F, {}_a\overline{G}_0\right) \to H^1(F, \mathrm{PGL}(W))\right) / {}_a\Theta_0^{\Gamma, \rho_0}. $$

Proof. Let the notation be as before. It is enough to prove that if $(G, \rho_\theta, V_\theta)$ and $(G, \rho_{\theta'}, V_{\theta'})$ are equivalent to each other over F for $\theta, \theta' \in {}_a\Theta_0^{\Gamma}$, then $c\theta\theta_0 = c\theta'$ for $\theta_0 \in {}_a\Theta_0^{\Gamma, \rho_0}$. Under this condition, there exists an automorphism λ: $G \to G$ defined over F and $\rho_{\theta'} \circ \lambda$ is equivalent to ρ_θ. Let $\lambda = f_a \lambda f^{-1}$ with ${}_a\lambda \in \operatorname{Aut}({}_aG_0)$ and set ${}_a\lambda = \alpha\beta$, $\alpha \in \operatorname{Int}({}_aG_0)$, and $\beta \in {}_a\Theta_0$. Let $a_\sigma = f^{-1\,\sigma}f$. Then by the first condition we have

$$ a_\sigma {}^\sigma\alpha {}^\sigma\beta = \alpha\beta a_\sigma = \alpha\beta a_\sigma \beta^{-1}\beta, $$

since λ is defined over F. From this we obtain

$$ a_\sigma {}^\sigma\alpha = \alpha\beta a_\sigma \beta^{-1}, \qquad {}^\sigma\beta = \beta. $$

This implies $\beta \in {}_a\Theta_0^{\Gamma}$ and $c\beta^{-1} = c$. Now we see ${}_a\rho_{\theta'} \circ \lambda$ is equivalent to ${}_a\rho_0 \circ \theta'^{-1} \circ \alpha \circ \beta \circ f^{-1}$, hence to ${}_a\rho_0 \circ \theta'^{-1} \circ \beta \circ f^{-1}$. By the second condition we see ${}_a\rho_0 \circ \theta'^{-1} \circ \beta$ is equivalent to ${}_a\rho_0 \circ \theta^{-1}$. This shows $\theta'^{-1}\beta\theta = \theta_0 \in {}_a\Theta_0^{\Gamma, \rho_0}$ and $c\theta'\theta_0 = c\theta$. This completes the proof.

To check the triviality of ${}_a\rho_{0*}(c)$ in $H^1(F, \mathrm{PGL}(W))$, we consider the commutative diagram

$$ \begin{array}{ccc} H1\left(F, {}_a\overline{G}_0\right) & \xrightarrow{{}_a\rho_{0*}} & H^1(F, \mathrm{PGL}(W)) \\ \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow \\ H^2(F, Z({}_aG_0)) & \xrightarrow{{}_a\rho'_{0*}} & H^2(F, \mathbf{G}_m) \end{array} \tag{1.2} $$

induced by the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z({}_a G_0) & \longrightarrow & {}_a G_0 & \longrightarrow & {}_a \overline{G}_0 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & GL(W) & \longrightarrow & PGL(W) & \longrightarrow & 1. \end{array}$$

In (1.2), the second vertical is injective. Let δ be the first vertical map. Then ${}_a\rho_{0*}(c)$ is trivial if and only if ${}_a\rho'_{0*}(\delta(c))$ is trivial.

PROPOSITION 1.3. *Let the notation be as in Theorem* 1.2. *Let φ be the composite of δ and the canonical map of $H^2(F, Z({}_aG_0))$ to $H^2(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0)$. Then $\rho = {}_a\rho_0 \circ f^{-1}$ is equivalent to a representation defined over F if and only if $\varphi(c)$ is trivial. Let ι be the natural map $H^2(F, \mathrm{Ker}\,{}_a\rho_0) \to H^2(F, Z({}_aG_0))$. Then* $\mathrm{Ker}\,{}_a\rho_{0*}$ *is in one-to-one correspondence with*

$$\delta^{-1}\big(\iota\big(H^2(F, \mathrm{Ker}\,{}_a\rho_0)\big)\big)$$

and with

$$H^1(F, {}_aG_0/\mathrm{Ker}\,{}_a\rho_0)/H^1\big(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0\big).$$

Proof. We note $Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0$ is isomorphic to a finite group in $\mathbf{G}_m$, the center of $GL(W)$, hence a cyclic group. Let n be its order. We consider the exact sequence

$$1 \to Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0 \to \mathbf{G}_m \to \mathbf{G}_m \longrightarrow 1,$$

where the map $\mathbf{G}_m \to \mathbf{G}_m$ is given by $a \to a^n$. Then we have the exact sequence

$$H^1(F, \mathbf{G}_m) \to H^2\big(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0\big) \to H^2(F, \mathbf{G}_m)$$

associated with it. From this we see the map of $H^2(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0)$ to $H^2(F, \mathbf{G}_m)$ is injective. Since ${}_a\rho'_{0*}$ factors through $H^2(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0)$, the first assertion is clear.

The second assertion can be seen easily, since the kernel of the natural map

$$H^2\big(F, Z({}_aG_0)\big) \to H^2\big(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0\big)$$

is $\iota(H^2(F, \mathrm{Ker}\,{}_a\rho_0))$. The last assertion follows from the exact sequence

$$H^1(F, Z({}_aG_0)/\mathrm{Ker}\,\rho_0) \to H^1(F, {}_aG_0/\mathrm{Ker}\,{}_a\rho_0) \to H^1(F, {}_a\overline{G}_0) \to H^2(F, Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0)$$

associated with

$$1 \to Z({}_aG_0)/\mathrm{Ker}\,{}_a\rho_0 \to {}_aG_0/\mathrm{Ker}\,{}_a\rho_0 \to {}_a\overline{G}_0 \to 1.$$

Remark 1.4. In Proposition 1.1, it can be shown (cf. [B-T], Sect. 12.6, 7, and [T1]) more generally that if D is the division algebra over F determined by ${}_a\rho_{0*}(c) \in H^1(F, \mathrm{PGL}(W))$, then ${}_a\rho_0 \circ f^{-1}$ is equivalent to a representation of G into $\mathrm{GL}_{n/d}(D)$ defined over F, where $\dim W = n$ and d is the index of D. We know by the preceding argument that the class ${}_a\rho_{0*}(c)$ is determined by $\varphi(c)$.

In the next section, we determine F-forms of reduced irreducible reduced prehomogeneous vector spaces. We give a list of their semisimple parts and $\mathrm{Ker}\,\rho_0$, the numbering of which is due to [S-K]. We restrict I(1) to the case where $G_0 = \mathrm{SL}_n \times \mathrm{SL}_n$ and omit III(1) for simplicity. We follow the notation of [S-K]. In particular, Λ_i are the fundamental dominant weights of G_0, $1 \le i \le l$, for $l = \mathrm{rank}\, G_0$, and $V(n)$ is an n-dimensional vector space. We denote by μ_n the groups of nth roots of unity and set $\Delta(\mu_n) = \{(a, a^{-1}) | a \in \mu_n\}$. The asterisk indicates it is of parabolic type. In I(15), ψ is a nontrivial homomorphism of Z to μ_2 with $Z = \mu_4$ if $4 \nmid m$ and $Z = \mu_2 \times \mu_2$ if $4|m$.

I. A regular prehomogeneous vector space

(1)* $(\mathrm{SL}_n \times \mathrm{SL}_n,\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(n))$, $\Delta(\mu_n)$.
(2)* $(\mathrm{SL}_n, 2\Lambda_1, V(n(n+1)/2))$, $(n \ge 2)$, μ_1 (n odd), μ_2 (n even).
(3)* $(\mathrm{SL}_{2m}, \Lambda_2, V(m(2m-1)))$, $(m \ge 3)$, μ_2.
(4)* $(\mathrm{SL}_2, 3\Lambda_1, V(4))$, μ_1.
(5)* $(\mathrm{SL}_6, \Lambda_3, V(20))$, μ_3.
(6)* $(\mathrm{SL}_7, \Lambda_3, V(35))$, μ_1.
(7)* $(\mathrm{SL}_8, \Lambda_3, V(56))$, μ_1.
(8)* $(\mathrm{SL}_3 \times \mathrm{SL}_2,\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2))$, μ_1.
(9)* $(\mathrm{SL}_6 \times \mathrm{SL}_2,\ \Lambda_2 \otimes \Lambda_1,\ V(15) \otimes V(2))$, $\mu_2 \times \mu_1$.
(10)* $(\mathrm{SL}_5 \times \mathrm{SL}_3,\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$, μ_1.
(11)* $(\mathrm{SL}_5 \times \mathrm{SL}_4,\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4))$, μ_1.
(12)* $(\mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{SL}_2,\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3) \otimes V(2))$, $\Delta(\mu_3) \times \mu_1$.
(13)* $(\mathrm{Sp}_{2n} \times \mathrm{SL}_{2m},\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m))$ $(n \ge 2m \ge 2)$, $\Delta(\mu_2)$.

(14)* $(\mathrm{Sp}_6, \Lambda_3, V(14))$, μ_1.
(15)* $(\mathrm{Spin}_n \times \mathrm{SL}_m,\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$, $(n \geq 3,\ n/2 \geq m \geq 1)$,
$\mu_2 \times \mu_1$ (n or m odd), $\{(a, \psi(a)) | a \in Z\}$ (otherwise).
(16)* $(\mathrm{Spin}_7$, spin rep., $V(8))$, μ_1.
(17) $(\mathrm{Spin}_7 \times \mathrm{SL}_2$, spin rep.$\otimes \Lambda_1$, $V(8) \otimes V(2))$, $\Delta(\mu_2)$.
(18) $(\mathrm{Spin}_7 \times \mathrm{SL}_3$, spin rep.$\otimes \Lambda_1$, $V(8) \otimes V(3))$, μ_1.
(19) $(\mathrm{Spin}_9$, spin rep., $V(16))$, μ_1.
(20)* $(\mathrm{Spin}_{10} \times \mathrm{SL}_2$, half-spin rep.$\otimes \Lambda_1$, $\mathrm{V}(16) \otimes \mathrm{V}(2))$, $\Delta(\mu_2)$.
(21)* $(\mathrm{Spin}_{10} \times \mathrm{SL}_3$, half-spin rep.$\otimes \Lambda_1$, $V(16) \otimes V(3))$, μ_1.
(22) $(\mathrm{Spin}_{11}$, spin rep., $V(32))$, μ_1.
(23)* $(\mathrm{Spin}_{12}$, half-spin rep., $V(32))$, μ_2.
(24)* $(\mathrm{Spin}_{14}$, half-spin rep., $V(64))$, μ_1.
(25) $(G_2, \Lambda_1, V(7))$, μ_1.
(26) $(G_2 \times \mathrm{SL}_2,\ \Lambda_1 \otimes \Lambda_1,\ V(7) \otimes V(2))$, μ_1.
(27)* $(E_6, \Lambda_1, V(27))$, μ_1.
(28)* $(E_6 \times \mathrm{SL}_2,\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$, μ_1.
(29)* $(E_7, \Lambda_1, V(56))$, μ_1.

II. A nonregular prehomogeneous vector space, with relative invariants

$(\mathrm{Sp_n} \times \mathrm{Spin}_3,\ \Lambda_1 \otimes \Lambda_1,\ \mathrm{V(2n)} \otimes \mathrm{V}(3))$, $\mu_1 \times \mu_2$.

III. A nonregular prehomogeneous vector space, without relative invariants

(2) $(\mathrm{SL}_n \times \mathrm{SL}_m,\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m))$ $(m/2 \geq n \geq 1)$, $\Delta(\mu_{(n,m)})$.
(3) $(\mathrm{SL}_{2m+1},\ \Lambda_2,\ V(m(2m+1)))$ $(m \geq 2)$, μ_1.
(4) $(\mathrm{SL}_{2m+1} \times \mathrm{SL}_2,\ \Lambda_2 \otimes \Lambda_1,\ V(m(2m+1)) \otimes V(2))$ $(m \geq 2)$, μ_1.
(5) $(\mathrm{Sp}_n \times \mathrm{SL}_{2m+1},\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))$ $(n > 2m+1 \geq 1)$, μ_1.
(6) $(\mathrm{Spin}_{10}$, half-spin rep., $V(16))$, μ_1.

Remark 1.5. We remark on castling transformation and classification. Let (H, ρ, V) be a triplet of a quasisplit semisimple simply connected group and its irreducible representation ρ on V and let ρ^* be the contragredient representation of ρ on the dual space V^* of V. Let us consider the triplets

$$\left(G_p, \rho_p, \tilde{V}_p\right) = \left(H \times \mathrm{SL}_p,\ \rho \otimes \Lambda_1,\ V \otimes V(p)\right),$$

$$\left(G_q, \rho_q, \tilde{V}_q\right) = \left(H \times \mathrm{SL}_q,\ \rho^* \otimes \Lambda_1,\ V^* \otimes V(q)\right)$$

for $p + q = \dim V$. Then $(G_p, \rho_p, \tilde{V}_p)$ is the semisimple part of a prehomogeneous vector space if and only if $(G_q, \rho_q, \tilde{V}_q)$ is so, and then $(\tilde{G}_p, \tilde{\rho}_p, \tilde{V}_p)$ is called a castling transform of $(\tilde{G}_q, \tilde{\rho}_q, \tilde{V}_q)$. It is easy to see $\operatorname{Ker} \rho_p \simeq \operatorname{Ker} \rho_q$. Assume neither $(\mathrm{SL}_p, \Lambda_1)$ nor $(\mathrm{SL}_q, \Lambda_1)$ appear in the tensor product decomposition of (H, ρ) and the condition (1.1) is satisfied for (H, ρ). Then the F-form of G_i is of the form $H' \times H_i''$ for $i = p, q$, where H' is an F-form of H, H_i'' is an F-form of SL_i, and ${}_a\Theta_0^{\Gamma, \rho_0}$s for G_i coincide with each other in the sense that the elements in that group induce the identity on H_i'' and are identical on H'. An easy consideration shows

$$\delta^{-1}\Big(\iota\Big(H^2\big(F, \operatorname{Ker} \rho_p\big)\Big)\Big) \simeq \delta^{-1}\Big(\iota\Big(H^2\big(F, \operatorname{Ker} \rho_q\big)\Big)\Big).$$

The condition (1.1) holds also for $(G_i, \rho_i, \tilde{V}_i)$ for $i = p, q$. Hence the classification of F-forms of $(G_p, \rho_p, \tilde{V}_p)$ is identical to that of $(G_q, \rho_q, \tilde{V}_q)$.

2. CLASSIFICATION OF PREHOMOGENEOUS VECTOR SPACES

In this section, we classify prehomogeneous vector spaces calculating Galois cohomology in Theorem 1.2 and Proposition 1.3 case by case. We usually give a proof for F an algebraic number field, since the case of local fields is easier. Let (G, ρ, V) be a triplet of a connected semisimple simply connected algebraic group G, a representation ρ of G on a vector space V defined over F such that $(G \times \mathbf{G}_m, \rho \otimes 1, V)$ is an F-form of a reduced irreducible regular prehomogeneous vector space. Let (G_0, ρ_0, V_0) be one of the triplets given in the list of the previous section. We call this a split form. By the list in Section 1 and considering the action of Γ on Δ, we see easily formulate the following lemma.

LEMMA 2.1. *Let the notation be as in Section* 1 *and the preceding text.*

(1) $\operatorname{Ker} \rho \neq \mu_1$ *only in the cases of* I (1)–(3), (5), (9), (12), (13), (15), (17), (20), (23), II, *and* III(2).

(2) *The outer form can appear in the cases of* I(1), (5), (12), *and* (15).

(3) ${}_a\Theta_0^{\Gamma, \rho} \neq \{1\}$ *only in the cases of* I(1), (5), (12), *and* (15) *for n even and m even*.

(4) *The condition* (1.1) *is satisfied in all cases*.

Our classification depends on the types of $\operatorname{Ker} \rho$ and whether outer forms appear. First we treat the cases where $\operatorname{Ker} \rho = \mu_1$, that is, the cases of I(4), (6)–(8), (10), (11), (14), (16), (18), (19), (21), (22), (24)–(29), III(3)–(6) in Theorem 2.2. Then we treat the cases where $\operatorname{Ker} \rho \neq \mu_1$ and only inner forms of split groups appear, that is, the cases of I(2), (3), (9), (13), (17), (20), (23), II, and III(2) in Theorems 2.4, 2.5, and 2.6. Last, we treat the cases where $\operatorname{Ker}\rho \neq \mu_1$ and outer forms appear, that is, the cases of I(1), (5), (12), and (15) in Theorems 2.7, 2.9, and 2.11.

Before proceeding to the calculation, we recall some results on Galois cohomology and prepare some notation. Let F be an algebraic number field or a local field of characteristic 0. For an algebraic number field F, we denote by Σ the set of all places of F, and by Σ_∞ the set of all infinite places of F. For $v \in \Sigma$, let F_v be the completion of F at v. From the exact sequence

$$1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1$$

we obtain a map

$$H^1(F, \mathrm{PGL}_n) \to H^2(F, \mu_n) \tag{2.1}$$

and we know this is bijective for F as before. For a non-archimedean local F, we have a canonical isomorphism

$$H^2(F, \mu_n) \simeq \tfrac{1}{n}\mathbf{Z}/\mathbf{Z}, \tag{2.2}$$

and for $\mathbf{R}$

$$H^2(F, \mu_n) \simeq \begin{cases} \frac{1}{2}\mathbf{Z}/\mathbf{Z}, & \text{if } n \text{ is even}, \\ \mu_1, & \text{if } n \text{ is odd}. \end{cases}$$

We denote by inv the composition of (2.1) and (2.2). When F is an algebraic number field and v is a place of F, we denote by inv_v the foregoing map for F_v.

Let $S_n(F)$ be the space of symmetric matrices of degree n over F and let $\mathrm{Alt}_n(F)$ be the space of antisymmetric matrices of degree n over F. For a quadratic extension K of F with the nontrivial automorphism ι of K over F, or a central simple algebra B over K with an involution ι of the first or the second kind, let $H_n(K)$ and $H_n(B)$ denote the space of hermitian matrices of degree n with coefficients in K or B with respect to ι. For a division quaternion algebra D over F, let $AH_n(D)$ be the space of anti-hermitian matrices of degree n with coefficients in D with respect to the canonical involution ι. For $q \in S_n(F)$, let $\mathrm{disc}(q)$ be the class of $(-1)^{n(n+1)/2}\det q$ in $F/F^{\times 2}$, and for $h \in AH_n(D)$, let $\mathrm{disc}(h)$ be the class of $(-1)^n N(h)$ in $F/F^{\times 2}$, where N is the reduced norm of $M_n(D)$. For $h \in H_n(K)$, let $\mathrm{disc}(h)$ be the class of $\det h \in H/N_{K/F}(K^\times)$. For $q \in S_n(F)$, let $\mathrm{O}(q)$ and $\mathrm{SO}(q)$ be the orthogonal and the special orthogonal groups of q, respectively, and let $\mathrm{Spin}(q)$ be the universal covering group of $\mathrm{SO}(q)$. For $h \in H_n(K)$, $H_n(B)$, or $AH_n(D)$, let $\mathrm{U}(h)$ and $\mathrm{SU}(h)$ be the unitary and the special unitary groups of h. For $h \in AH_n(D)$, let $\mathrm{Spin}(h)$ be the universal covering group of $\mathrm{SU}(h)$.

Let $q \in S_n(F)$ with $\operatorname{disc}(q) \neq 0$. If F is a local field, we denote by $\varepsilon(q)$ the Hasse invariant of q defined as follows. For x, choose $g \in \mathrm{GL}_n(F)$ so that ${}^t gxg$ is a diagonal matrix with the diagonal components $a_1, a_2, \dots, a_n$, and set

$$\varepsilon(q) = \prod_{i \leq j} (a_i, a_j),$$

where $(\,,)$ is the Hilbert symbol in F. When F is an algebraic number field, let $\varepsilon_v(q)$ be the Hasse invariant of q considered as an element of $S_n(F_v)$.

For a fixed $q_0 \in S_n(F)$ with $\operatorname{disc}(q_0) \neq 0$, it is known that $H^1(F, \mathrm{O}(q_0))$ classifies $f \in S_n(F)$ with $\operatorname{disc}(q) \neq 0$ over F and that $H^1(F, \mu_2) \simeq F^\times / F^{\times 2}$. The map of $H^1(F, \mathrm{O}(q_0))$ to $H^1(F, \mu_2)$ induced by the exact sequence

$$1 \to \mathrm{SO}(q_0) \to \mathrm{O}(q_0) \to \mu_2 \to 1$$

coincides with $\operatorname{disc}(q)/\operatorname{disc}(q_0)$ under this identification. Moreover, it is known that $H^1(F, \mathrm{SO}(q_0))$ classifies $q \in S_n(F)$ with $\operatorname{disc}(q) = \operatorname{disc}(q_0)$. From the exact sequence

$$1 \to \mu_2 \to \mathrm{Spin}(q_0) \to \mathrm{SO}(q_0) \to 1$$

we obtain the coboundary map

$$H^1(F, \mathrm{SO}(q_0)) \to H^2(F, \mu_2).$$

For F a local field, this is given by $c \mapsto \varepsilon(q)/\varepsilon(q_0)$ (cf. [Sp]), for $c \in H^1(F, \mathrm{SO}(q_0))$, where q is an element of $S_n(F)$ corresponding to c. Here $H^2(F, \mu_2)$ is identified with μ_2.

A similar invariant $c(h_0, h)$ was defined for $h \in AH_n(D)$ in [B1, B2] for a quaternion algebra D over an algebraic number field F. As in the case of symmetric matrices, $H^1(F, U(h_0))$ for $h_0 \in AH_n(D)$ with $\operatorname{disc}(h_0) \neq 0$ classifies $h \in AH_n(D)$ with $\operatorname{disc}(h) \neq 0$, and the map η in

$$H^1(F, \mathrm{SU}(h_0)) \xrightarrow{\tau} H^1(F, \mathrm{U}(h_0)) \xrightarrow{\eta} H^1(F, \mu_2)$$

induced by

$$1 \to \mathrm{SU}(h_0) \to U(h_0) \to \mu_2 \to 1$$

coincides with $\operatorname{disc}(h)/\operatorname{disc}(h_0)$. Let $c \in H^1(F, \mathrm{U}(h_0))$ be the class corresponding to h and assume $\eta(c) = 1$. Then $\tau^{-1}(c)$ consists of two classes c_1, c_2. Let Δ be the coboundary map of $H^1(F, \mathrm{SU}(h_0))$ to $H^2(F, \mu_2)$ induced by

$$1 \to \mu_2 \to \mathrm{Spin}(h_0) \to \mathrm{SU}(h_0) \to 1$$

and let $c(D)$ be the class in $H^2(F, \mu_2)$ corresponding to D. Then $\Delta(c_1)\Delta(c_2)^{-1} = c(D)$ and the class $c(h_0, h)$ is defined as the class of $\Delta(c_1)$ in $H^2(F, \mu_2)/\langle c(D)\rangle$.

In the rest of this paper, we denote by q_0 the symmetric matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

if $n = 2m$, and

$$\begin{pmatrix} 0 & E_m & 0 \\ E_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

if $n = 2m + 1$, where E_m is the unit matrix of degree m. These matrices are denoted by h_0 when considered as elements of $H_n(K)$. For a division quaternion algebra D over F and an even integer n, let h_0 be an element of $AH_n(D)$ with maximal rank, that is,

$$\begin{pmatrix} 0 & E_{n/2} \\ -E_{n/2} & 0 \end{pmatrix}.$$

We begin with the cases where $\operatorname{Ker}\rho = \mu_1$.

THEOREM 2.2. *Let (G, ρ, V) be of type* I(4), (6)–(8), (10), (11), (14), (16), (18), (19), (21), (22), (24)–(29) *and* III(3)–(6). *If F is an algebraic number field, we have*

$$\operatorname{Ker}\rho_{0*} \simeq \prod_{v\in\Sigma_\infty} \operatorname{Ker}\rho_{0v*},$$

where ρ_{0v*}: $H^1(F_v, \overline{G}_0) \to H^1(F_v, \mathrm{PGL}(V))$ induced by ρ_{0v}, the representation ρ_0 over F_v. Namely, the classification is reduced to that over **R**. *In particular, in the cases of* (4), (6)–(8), (10), (11), (14) *and* III(3)–(5), *there exist only the split forms.*

If F is a non-archimedean local field, one has only the split forms.

Proof. In the preceding cases, it is enough to consider inner forms of the split group G_0. Since $\operatorname{Ker}\rho_0 = \mu_1$ and $\Theta_0^{\Gamma,\rho_0} = \mu_1$, it is enough to determine Ker ρ_{0*}, hence Ker δ, by Proposition 1.3, where δ is the coboundary map $H^1(F, \overline{G}_0) \to H^2(F, Z(G_0))$ associated with the exact sequence

$$1 \to Z(G_0) \to G_0 \to \overline{G}_0 \to 1.$$

From this, we obtain the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{F}, \mathrm{G}_0) \arrow[r, "\pi"] \arrow[d, "\simeq"] & \mathrm{H}^1(\mathrm{F}, \overline{\mathrm{G}}_0) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{G}_0)) \arrow[d] \\
\prod_{v \in \Sigma_\infty} \mathrm{H}^1(\mathrm{F}_v, \mathrm{G}_0) \arrow[r, "\prod \pi_v"] & \prod_{v \in \Sigma} \mathrm{H}^1(\mathrm{F}_v, \overline{\mathrm{G}}_0) \arrow[r] & \prod_{v \in \Sigma} \mathrm{H}^2(\mathrm{F}_v, \mathrm{Z}(\mathrm{G}_0)).
\end{tikzcd}$$

Here the horizontal sequences are exact, the vertical maps are injective, and the first vertical map is bijective (cf. [P-R], Theorems 6.6 and 6.22). Then by the foregoing diagram, we see easily Ker $\rho_{0*} = \mathrm{Im}\,\pi$, and it is in one-to-one correspondence with

$$\prod_{v \in \Sigma_\infty} \pi_v\big(H^1(F_v, G_0)\big)$$

and with $\prod_{v \in \Sigma_\infty} \mathrm{Ker}\,\rho_{0v*}$. This proves the first assertion. As for the second one, we note in the cases indicated in the assertion, $H^1(F_v, G_0)$ vanishes and Ker ρ_{0*} consists only of the trivial class. This completes the proof.

In the preceding types, I(4), (6)–(8), (10), (11), (14), (16), (21), (24), (27), and (28) are of parabolic type and their classification was completed in [R]. The other cases can be treated easily by Galois cohomology.

PROPOSITION 2.3. *Let $F = \mathbf{R}$.*

(1) *Let (G, ρ, V) be of type* I(18). *Then $G \simeq \mathrm{Spin}(q) \times \mathrm{SL}_3$ with $q = q_0$ or $-E_7$.*

(2) *Let (G, ρ, V) be of type* I(19) *or* (22). *Then $G \simeq \mathrm{Spin}(q)$ with $q = q_0$, E_9, or $\begin{pmatrix} 1 & 0 \\ 0 & -E_8 \end{pmatrix}$ for* (19), *and $q = q_0$, $\begin{pmatrix} E_2 & 0 \\ 0 & -E_9 \end{pmatrix}$ or $\begin{pmatrix} E_{10} & 0 \\ 0 & -1 \end{pmatrix}$ for* (22).

(3) *Let (G, ρ, V) be of type* I(25) *and* (26). *Then $G \simeq H_1 \times H_2$, where H_1 is one of the two groups of type G_2 over $\mathbf{R}$ and $H_2 = \mu_1$ for* (25) *and* SL_2 *for* (26).

(4) *Let (G, ρ, V) be of type* III(6). *Then $G \simeq \mathrm{Spin}(q)$ for $q = q_0$ or* $\begin{pmatrix} E_9 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. Assertions (1) and (2) easily follow from the fact that the coboundary map $H^1(\mathbf{R}, \overline{G}_0) \to H^2(\mathbf{R}, Z(G_0))$ is injective for $G_0 = \mathrm{SL}_n$ and is given by the Hasse invariant for $G_0(= \overline{G}_0) = \mathrm{SO}(q_0)$ with n odd.

The third assertion follows from this and the fact that $Z(H) = \mu_1$ for a group H of type G_2 and there exist two isomorphism classes of algebraic groups of type G_2 over $\boldsymbol{R}$. For (4), let $Y \simeq \mu_2$ be the kernel of the map of $\mathrm{Spin}(q_0)$ to $\mathrm{SO}(q_0)$. From the diagram

$$\begin{tikzcd}
1 \arrow[r] & Z(\mathrm{Spin}(q_0)) \arrow[r] \arrow[d] & \mathrm{Spin}(q_0) \arrow[r] \arrow[d] & \overline{\mathrm{Spin}(q_0)} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{SO}(q_0) \arrow[r] & \overline{\mathrm{SO}(q_0)} \arrow[r] & 1,
\end{tikzcd}$$

we obtain

$$\begin{tikzcd}
H^1(\boldsymbol{R}, \mathrm{Spin}(q_0)) \arrow[r] \arrow[d] & H^1(\boldsymbol{R}, \overline{\mathrm{Spin}(q_0)}) \arrow[r, "\delta"] \arrow[d, equal] & H^2(\boldsymbol{R}, Z(\mathrm{Spin}(q_0))) \arrow[d] \\
H^1(\boldsymbol{R}, \mathrm{SO}(q_0)) \arrow[r] \arrow[d, "\varepsilon"] & H^1(\boldsymbol{R}, \overline{\mathrm{SO}(q_0)}) \arrow[r] & H^2(\boldsymbol{R}, \mu_2) \\
H^2(\boldsymbol{R}, Y), & &
\end{tikzcd}$$

where two rows and the first column are exact and ε is the Hasse invariant. From this, we see the kernel of δ comes from classes $c \in H^1(\boldsymbol{R}, \mathrm{SO}(q_0))$ such that $\varepsilon(c) = 1$. This completes the proof.

Next we treat the cases of I(2), (3), (9), (13), (17), (20), (23), II, and III(2), where there appear only inner forms of split groups and Ker $\rho \neq \mu_1$. First we treat the cases of I(2), (3), (9), (13), II, and III(2). In these cases, we have $\Theta_0^{\Gamma, \rho_0} = \mu_1$.

THEOREM 2.4. *Let F be an algebraic number field or a local field.*

(1) *Let* (G, ρ, V) *be of type* I(2). *Then* (G, ρ, V) *is equivalent to* (a) $(\mathrm{SL}_n, \bar{\rho}, S_n(F))$, *where*

$$\bar{\rho}(g)v = gv\,{}^tg, \tag{2.3}$$

or to (b) $(\mathrm{SL}_{n/2}(D), \bar{\rho}, AH_{n/2}(D))$ *for an even integer n and a division quaternion algebra D with the canonical involution* ι, *where*

$$\bar{\rho}(g)v = gv\tilde{g} \tag{2.4}$$

and $\tilde{g} = {}^t({}^\iota g_{ij})$ *for* $g = (g_{ij})$.

(2) *Let* (G, ρ, V) *be of type* I(3). *Then* (G, ρ, V) *is equivalent to* (a) $(\mathrm{SL}_n, \bar{\rho}, \mathrm{Alt}_n(F))$, *with* $\bar{\rho}$ *as in* (2.3), *or to* (b) $(\mathrm{SL}_{n/2}(D), \bar{\rho}, H_{n/2}(D))$ *for*

an even integer n and a division quaternion algebra D over F with $\bar{\rho}$ *as in* (2.4).

(3) *Let* (G, ρ, V) *be of type* I(9). *Then* (G, ρ, V) *is equivalent to* (a) $(\mathrm{SL}_6 \times \mathrm{SL}_2, \bar{\rho}, \mathrm{Alt}_6(F) \oplus \mathrm{Alt}_6(F))$, *where*

$$\bar{\rho}((g, 1))(v_1, v_2) = \left(gv_1{}^t g, gv_2{}^t g\right), \qquad g \in \mathrm{SL}_6,$$

$$\bar{\rho}((1, h))(v_1, v_2) = (v_1, v_2)^t h, \qquad h \in \mathrm{SL}_2,$$

or to (b) $(\mathrm{SL}_3(D) \times \mathrm{SL}_2, \bar{\rho}, H_3(D) \oplus H_3(D))$ *for a division quaternion algebra D, where*

$$\bar{\rho}((g, 1))(v_1, v_2) = \left(gv_1 \tilde{g}, gv_2 \tilde{g}\right), \qquad g \in \mathrm{SL}_3(D),$$

$$\bar{\rho}((1, h))(v_1, v_2) = (v_1, v_2)^t h, \qquad h \in \mathrm{SL}_2,$$

with $\tilde{g}$ *as in* (1)(b).

(4) *Let* (G, ρ, V) *be of type* I(13). *Then* (G, ρ, V) *is equivalent to* (a) $(\mathrm{Sp}_{2n} \times \mathrm{SL}_{2m}, \bar{\rho}, M_{2n\, 2m}(F))$, *where*

$$\bar{\rho}((g_1, g_2))v = g_1 v^t g_2, \qquad g_1 \in \mathrm{Sp}_{2n},\ g_2 \in \mathrm{SL}_{2m},$$

or to (b) $(\mathrm{SU}_n(D) \times \mathrm{SL}_m(D), \bar{\rho}, M_{n/2\, m/2}(D))$ *for a division quaternion algebra D over F, where*

$$\bar{\rho}((g_1, g_2))v = g_1 v \tilde{g}_2, \qquad g_1 \in \mathrm{SU}_n(D),\ g_2 \in \mathrm{SL}_m(D),$$

with $\tilde{g}_2$ *as in* (1)(b).

(5) *Let* (G, ρ, V) *be of type* II. *Then* (G, ρ, V) *is equivalent to* $(\mathrm{Sp}_n \times \mathrm{SL}_1(D), \bar{\rho}, F^n \otimes D^0)$, *where D is a quaternion algebra over F* (*split or nonsplit*), $D^0 = \{x \in D | \mathrm{Tr}\, x = 0\}$, *and*

$$\bar{\rho}((g_1, g_2))v_1 \otimes v_2 = g_1 v_1 \otimes g_2 v_2{}^{\iota} g_2.$$

Here Tr *denotes the reduced trace of D*.

(6) *Let* (G, ρ, V) *be of type* III(2). *Then* (G, ρ, V) *is equivalent to*

$$\left(\mathrm{SL}_{n/l}(B) \times \mathrm{SL}_{m/l}(B^0), \bar{\rho}, M_{n/l\, m/l}(B)\right),$$

where l is a divisor of (n, m), *B is a division algebra over F with* $\dim_F B = l^2$, B^0 *is its opposite algebra with an anti-isomorphism* ι *of* B^0 *to B, and* $\bar{\rho}$ *is as in* (2.4).

Proof. In these cases, $\mathrm{Ker}\, \rho_0 \simeq \mu_2$ except in the cases where (G, ρ, V) is of type I(2) and n is odd, or of type III(2). In the case of I(2) with n odd, $\mathrm{Ker}\, \rho_0 = \mu_1$ and can be treated as in Theorem 2.2.

Let us consider the other cases. Since the map $H^2(F, \operatorname{Ker}\rho_0) \to H^2(F, Z(G_0))$ is injective in these cases, it is enough to determine $\delta^{-1}(H^2(F, \operatorname{Ker}\rho_0))$ by Proposition 1.3. Take the case I(2). In this case, a class of $H^2(F, \operatorname{Ker}\rho_0)$ corresponds to a quaternion algebra D over F and we have $G = \mathrm{SL}_n(F)$ or $\mathrm{SL}_{n/2}(D)$ for a division quaternion algebra D over F. It is easy to see that the triplet (a) and (b) define F-forms of type I(2). Hence (a) and (b) give F-forms. The type I(3) can be treated in the same way.

Now take the type I(9). Let $G_0 = G_1 \times G_2$ with $G_1 = \mathrm{SL}_6$ and $G_2 = \mathrm{SL}_2$. Then $\operatorname{Ker}\rho_0 = \mu_2 \times \mu_1$ and $H^2(F, \operatorname{Ker}\rho_0) = H^2(F, \mu_2) \times \mu_1$. Our assertion follows from this as before. The case of II can be proved in the same way.

In the case of type I(13), let $G_0 = G_1 \times G_2$, where $G_1 = \mathrm{Sp}_{2n}$ and $G_2 = \mathrm{SL}_{2m}$. Then $\operatorname{Ker}\rho_0 = \{(a, a) | a \in \mu_2\}$ and $H^2(F, \operatorname{Ker}\rho_0) \simeq H^2(F, \mu_2)$ considered as a subgroup $H^2(F, Z(G_0)) = H^2(F, \mu_2) \times H^2(F, \mu_2)$ by the diagonal embedding. Our assertion follows from this as before.

For III(2), we have $\operatorname{Ker}\rho \simeq \mu_{(n,m)}$ and we can treat this case also in the same way as before. This completes the proof.

Next we treat the cases I(17) and (20) in Theorem 2.5 and the case I(23) in Theorem 2.6.

THEOREM 2.5. *Let F be an algebraic number field or a local field.*

(1) *Let (G, ρ, V) be of type* I(17). *Then (G, ρ, V) is equivalent to*

$$\left(\mathrm{Spin}(q) \times \mathrm{SL}_1(D), \bar{\rho}, D^2\right),$$

where $q \in S_7(F)$ with $\operatorname{disc}(q) = \operatorname{disc}(q_0)$ *and D is a quaternion algebra over F satisfying* $\varepsilon(q)/\varepsilon(q_0) = \operatorname{inv}(D)$ *if F is a local field and* $\varepsilon_v(q)/\varepsilon_v(q_0) = \operatorname{inv}_v(D)$ *for all $v \in \Sigma$ if F is an algebraic number field. $\bar{\rho}$ is given by*

$$\bar{\rho}((g_1, g_2))v = \rho_1(g_1)v^{\iota}g_2, \qquad (g_1, g_2) \in \mathrm{Spin}(q) \times \mathrm{SL}_1(D), \quad (2.5)$$

where ρ_1 is the spin representation of $\mathrm{Spin}(q)$ *into* $\mathrm{GL}_2(D)$.

(2) *Let (G, ρ, V) be of type* I(20). *Then (G, ρ, V) is equivalent to*

$$\left(\mathrm{Spin}(q) \times \mathrm{SL}_1(D), \rho, D^8\right),$$

where $q \in S_{10}(F)$ with $\operatorname{disc}(q) = \operatorname{disc}(q_0)$ *and D is a quaternion algebra over F satisfying the condition in* (1). *ρ is given by* (2.5), *where ρ_1 is a half-spin representation of* $\mathrm{Spin}(q)$ *into* $\mathrm{GL}_8(D)$.

Proof. In the case of I(17), $Z(G_0) \simeq \mu_2 \times \mu_2$, $\operatorname{Ker}\rho_0 \simeq \mu_2$, and $\operatorname{Ker}\rho_0$ is diagonally embedded into $Z(G_0)$. We see the map $H^2(F, \operatorname{Ker}\rho_0) \to$

$H^2(F, Z(G_0))$ is injective. Hence it is enough to determine $\delta^{-1}(H^2(F, \operatorname{Ker}\rho_0))$, where $H^2(F, \operatorname{Ker}\rho_0)$ is $H^2(F, \mu_2)$ considered as a subgroup of $H^2(F, \mu_2) \times H^2(F, \mu_2)$ by the diagonal embedding. Since the kernel of the half-spin representation ρ_1 is trivial, ρ_1 is realized in $\mathrm{GL}_2(D)$ over F by Remark 1.4. By the same argument as in Theorem 2.4(4) and the Hasse principle, we obtain the assertion.

In the case of I(20), let $H_1 = \mathrm{Spin}_{10}$ and $H_2 = \mathrm{SL}_2$. Then $Z(H_1) = \mu_4$ and $Z(H_2) = \mu_2$. If we consider μ_2 as a subgroup of μ_4 naturally, then we have an exact sequence

$$1 \to \operatorname{Ker}\rho_0 \to Z(H_1) \times Z(H_2) \to Z(H_1) \to 1,$$

where the map of $Z(H_1) \times Z(H_2)$ to $Z(H_1)$ is the multiplication in μ_4. From this, we see $\operatorname{Ker}\rho_0 = \{(a, a) | a \in \mu_2\}$ and the map $H^2(F, \operatorname{Ker}\rho_0) \to H^2(F, Z(G_0))$ is injective. Hence it is enough to determine $\delta^{-1}((\alpha, \alpha))$ in $H^1(F, \overline{G}_0)$ for each $\alpha \in H^1(F, \mu_2)$, since

$$\begin{aligned} H^2(F, \operatorname{Ker}\rho_0) &= \{(\alpha, \alpha) | \alpha \in H^2(F, \mu_2)\} \\ &\subset H^2(F, Z(H_1)) \times H^2(F, Z(H_2)). \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^1(\mathrm{F}, \mathrm{H}_1/\mathrm{Z}(\mathrm{H}_1)^2) & \longrightarrow & \mathrm{H}^1(\mathrm{F}, \overline{\mathrm{H}}_1) & \longrightarrow & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)/\mathrm{Z}(\mathrm{H}_1)^2) \\ \downarrow & & \downarrow{\scriptstyle \delta_1} & & \| \\ \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)^2) & \longrightarrow & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)) & \longrightarrow & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)/\mathrm{Z}(\mathrm{H}_1)^2), \end{array}$$

two rows of which are exact. The first vertical map is the Hasse invariant. Let δ_1 be the coboundary map of $H^1(F, \overline{H}_1)$ to $H^2(F, Z(H_1))$. Then the preceding diagram shows that if $c(\in H^1(F, \overline{H}_1)) \in \delta_1^{-1}(\alpha)$, then it is an image of $c' \in H^1(F, H_1/Z(H_1)^2) \simeq H^1(F, \mathrm{SO}(q_0))$ such that its image in $H^2(F, Z(H_1)^2) = H^2(F, \mu_2)$ gives α. Since the kernels of the half-spin representations are trivial, our assertion easily follows from this and Remark 1.4.

THEOREM 2.6. *Let F be an algebraic number field or a non-archimedean local field. Let (G, ρ, V) be of type* I(23). *Then (G, ρ, V) is equivalent to either*:

(a) $(\mathrm{Spin}(q), \bar{\rho}, V)$ *for $q \in S_{12}(F)$ such that* $\operatorname{disc}(q) = \operatorname{disc}(q_0)$, $\varepsilon(q) = \varepsilon(q_0)$ *if F is a local field, and* $\operatorname{disc}(q) = \operatorname{disc}(q_0)$ *and* $\varepsilon_v(q) = \varepsilon_v(q_0)$ *for all $v \in \Sigma$ if F is an algebraic number field. $\bar{\rho}$ is any one of the two half-spin representations of* $\mathrm{Spin}(q)$.

(b) $(\mathrm{Spin}(h), \bar{\rho}, V)$ *with $h \in AH_6(D)$ for a division quaternion algebra D over F such that* $\operatorname{disc}(h) = \operatorname{disc}(h_0)$ *if F is a local field and* $\operatorname{disc}(h) =$

disc(h_0), $c(h_0, h) = 1$ *if* F *is an algebraic number field*. $\bar{\rho}$ *is one of the half-spin representations of* $\mathrm{Spin}(h)$.

Proof. In this case it is enough to consider inner forms of G_0. Let Y be the kernel of the natural map of G_0 to $\mathrm{SO}(q_0)$. Then we have

$$Z(G_0) = \mathrm{Ker}\,\rho_0 \oplus Y$$

and $\mathrm{Ker}\,\rho_0 \simeq Y \simeq \mu_2$. From the exact sequence

$$1 \to Z(G_0)/Y \to \mathrm{SO}(q_0) \to \overline{G}_0 \to 1$$

we obtain the diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{F}, \mathrm{SO}(\mathrm{q}_0)) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathrm{F}, \overline{\mathrm{G}}_0) \arrow[r, "\eta"] \arrow[d, "\delta"] & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{G}_0)/\mathrm{Y}) \arrow[d, equal] \\
\mathrm{H}^2(\mathrm{F}, \mathrm{Y}) \arrow[r] \arrow[dr] & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{G}_0)) \arrow[r] \arrow[d] & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{G}_0)/\mathrm{Y}) \\
 & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{G}_0)/\mathrm{Ker}\,\rho_0). &
\end{tikzcd} \tag{2.6}$$

The first vertical map is the Hasse invariant. Since

$$H^2(F, Z(G_0)) = H^2(F, Y) \oplus H^2(F, \mathrm{Ker}\,\rho_0),$$

the kernel of the canonical map $H^2(F, Z(G_0)) \to H^2(F, Z(G_0)/\mathrm{Ker}\,\rho_0)$ is

$$\{(1, a) | a \in H^2(f, \mathrm{Ker}\,\rho_0)\}$$

and the map $H^2(F, \mathrm{Ker}\,\rho_0) \to H^2(F, Z(G_0))$ is injective. For each $a \in H^2(F, \mathrm{Ker}\,\rho_0) = H^2(F, \mu_2)$, we determine $\delta^{-1}((1, a))$. Let $c \in \delta^{-1}((1, a))$. Then $\eta(c) = a$ if we identify $\mathrm{Ker}\,\rho_0$ with $Z(G_0)/Y$.

If $a = 1$ in $H^2(F, \mathrm{Ker}\,\rho_0)$, then a class c in $\delta^{-1}((1, a))$ comes from an element of $H^1(F, \mathrm{SO}(q_0))$ whose coboundary class in $H^2(F, Y)$ is trivial. Hence we obtain the case (a). If $a \neq 1$ in $H^2(F, \mathrm{Ker}\,\rho_0) \simeq H^2(F, \mu_2)$, a determines a division quaternion algebra D over F. Let $h_0 \in AH_6(D)$ be of maximal index. Then by Proposition 12.10 of [B-T] and Propositions 7 and 8 of [T2], we know just one of the half-spin representations ρ' is defined over F for $\mathrm{Spin}(h_0)$. Let f be an isomorphism of $\mathrm{Spin}(h_0)$ to $\mathrm{Spin}(q_0)$ such that $f^{-1\,\sigma}f \in \mathrm{Int}(\mathrm{Spin}(q_0))$ for $\sigma \in \Gamma$ and $\rho_0 \circ f^{-1}$ is equivalent to ρ', and let c be the class of the 1-cocycle $(f^{-1\,\sigma}f)$ in $H^1(F, \overline{G}_0)$. Then we see c lies in $\delta^{-1}((1, a))$. Considering the twist by the 1-cocycle $(f^{-1\,\sigma}f)$, we see it is enough to determine $\mathrm{Ker}\,\delta'$, where δ' is the coboundary map

$$H^1\big(F, \overline{\mathrm{Spin}(h_0)}\big) \to H^2\big(F, Z(\mathrm{Spin}(h_0))\big).$$

By a similar diagram as (2.6), it is enough to determine the classes in $H^1(F, \mathrm{SU}(h_0))$ whose image in $H^2(F, {}_cY)$ by the coboundary map Δ in the definition of $c(h_0, h)$ is trivial. Let τ: $H^1(F, \mathrm{SU}(h_0)) \to H^1(F, U(h_0))$ be as before and for $c' \in H^1(F, \mathrm{SU}(h_0))$, let h be an element of $AH_6(D)$ corresponding to the class $\tau(c')$. Then it is easy to see if $\Delta(c') = 1$, then $c(h_0, h) = 1$ and if $c(h_0, h) = 1$, one of the classes c'' in $\tau^{-1}(\tau(c'))$ satisfies $\Delta(c'') = 1$. This proves our assertion.

Now we treat the cases where outer forms appear. First we determine the cases of I(1) and (12).

THEOREM 2.7. *Let F be an algebraic number field or a local field.*

(1) *Let (G, ρ, V) be of type* I(1). *Let F be an algebraic number field. Then (G, ρ, V) is equivalent to* (a) $(\mathrm{SL}_1(D) \times \mathrm{SL}_1(D^0), \bar{\rho}, D)$, *where D is a central simple algebra over F of* $\dim_F D = n^2$ *and D^0 is the opposite algebra of D with an anti-isomorphism ι from D^0 to D, and $\bar{\rho}$ is given by*

$$\bar{\rho}((g_1, g_2))v = g_1 v\,{}^{\iota}g_2,$$

or to (b) $(R_{K/F}(\mathrm{SL}_1(B)), \bar{\rho}, H_1(B))$, *where B is a central simple algebra over a quadratic extension K of F with an involution ι of the second kind and*

$$\bar{\rho}(g)v = gv\,{}^{\iota}g.$$

If F is a local field, then (G, ρ, V) is equivalent to (a) *for a central simple algebra D over F or* (b) *for $B = M_n(K)$ for a quadratic extension K of F.*

(2) *Let (G, ρ, V) be of type* I(12). *Let F be an algebraic number field. Then (G, ρ, V) is equivalent to* (a) $(\mathrm{SL}_1(D) \times \mathrm{SL}_1(D^0) \times \mathrm{SL}_2, \bar{\rho}, D \otimes F^2)$, *where D is a central simple algebra over F of* $\dim_F D = 3^2$, *D^0 and ι as before, and $\bar{\rho}$ is given by*

$$\bar{\rho}((g_1, g_2, g_3))(v_1 \otimes v_2) = g_1 v_1\,{}^{\iota}g_2 \otimes g_3 v_2,$$

or to (b) $(R_{K/F}(\mathrm{SL}_1(B)) \times \mathrm{SL}_2, \bar{\rho}, H_1(B) \otimes F^2)$, *where B is a central simple algebra of* $\dim_F B = 3^2$ *over a quadratic extension K of F with an involution ι of the second kind, and $\bar{\rho}$ is given by*

$$\bar{\rho}((g_1, g_2))(v_1 \otimes v_2) = g_1 v_1\,{}^{\iota}g \otimes g_2 v_2.$$

If F is a local field, (G,ρ,V) is equivalent to (a) *for a central simple algebra D over F of* $\dim_F D = 3^2$ *over F or* (b) *for $B = M_3(K)$ for a quadratic extension K of F.*

Proof. We will give a proof for (1). The assertion (2) can be proved in the same way. First assume G is isomorphic to $H_1 \times H_2$ over F, where H_i is an F-form of SL_n. We see easily outer forms do not appear as H_i. In this case, it is enough to consider inner forms of $G_0 = \mathrm{SL}_n \times \mathrm{SL}_n$. We see

$$\mathrm{Ker}\,\rho_0 = \left\{(a, a^{-1}) | a \in \mu_n\right\}.$$

Since $H^2(F, \mathrm{Ker}\,\rho_0) \to H^2(F, Z(G_0))$ is injective and

$$H^2(F, \mathrm{Ker}\,\rho_0) = \left\{(c, c^{-1}) | c \in H^2(F, \mu_n)\right\},$$

hence $H_1 \simeq \mathrm{SL}_1(D)$ and $H_2 \simeq \mathrm{SL}_1(D^0)$ for a central simple algebra D over F. We see easily $(\mathrm{SL}_1(D) \times \mathrm{SL}_1(D^0), \bar{\rho}, D)$ is of type I(1) and defined over F.

Now assume $G = R_{K/F}(H)$ for a quadratic extension K of F and an algebraic group H over K. As before, by considering the action of Γ, we see H is an inner form of SL_n. Hence ${}_aG_0 = R_{K/F}(\mathrm{SL}_n)$ and $Z({}_aG_0) = R_{K/F}(\mu_n)$. The map induced by ${}_a\rho_0$ is the norm map of $R_{K/F}(\mu_n)$ to $Z(\mathrm{GL}(V)) = \mu_n$, and $\mathrm{Ker}\,{}_a\rho_0 = R^{(1)}_{K/F}(\mu_n)$. Hence ${}_a\rho_0$ induces

$${}_a\rho_{0*}\colon H^2\big(F, R_{K/F}(\mu_n)\big) \simeq H^2(K, \mu_n) \to H^2(F, \mu_n).$$

The kernel of this map corresponds to a central simple algebra B over K with an involution ι of the second kind, and $H \simeq \mathrm{SL}_1(B)$. We see easily the prehomogeneous vector space given in (b) is of type I(1) and defined over F. The assertion in the case of local fields follows from the fact that there does not exist a nontrivial central simple algebra with an involution of the second kind.

Remark 2.8. In the case of (1)(a), $\Theta_0^{\Gamma,\rho_0} = \mu_2$ and the nontrivial element of Θ_0^{Γ,ρ_0} is given by the map $(g_1, g_2) \mapsto (g_2, g_1)$. It gives rise to an equivalence between $(\mathrm{SL}_1(D) \times \mathrm{SL}_1(D^0), \bar{\rho}, D)$ and $(\mathrm{SL}_1(D^0) \times \mathrm{SL}_1(D), \bar{\rho}^0, D^0)$ given by

$$(g_1, g_2) \mapsto (g_2, g_1), \qquad v_1 \mapsto {}^{\iota}v_1.$$

In the case of (1)(b), also ${}_a\Theta_0^{\Gamma,\rho_0} = \mu_2$, and the nontrivial element coincides with the action of the nontrivial element σ of $\mathrm{Gal}(K/F)$ on ${}_aG_0(F) = \mathrm{SL}_n(K)$. It gives an equivalence between the triplets for $\mathrm{SL}_1(B)$ and that for $\mathrm{SL}_1(B^0)$. Similar results hold also for type I(2).

Last we treat the case of I(5) in Theorem 2.9 and the case of I(15) in Theorem 2.11.

THEOREM 2.9. *Let* (G, ρ, V) *be of type* I(5). *Let* F *be an algebraic number field or a non-archimedean local field. Then* G *is isomorphic to either*:

(a) $\mathrm{SL}_2(D)$ *for a central simple algebra* D *over* F *of* $\dim_F D = 3^2$.

(b) $\mathrm{SU}(h)$ *for* $h \in H_1(B)$, *where* B *is a central simple algebra over a quadratic extension* K *over* F, *of index* 1 *or* 3 *of* $\dim_K B = 6^2$ *with an involution of the second kind, and* h *satisfies* $N(h)/\mathrm{disc}(h_0) \in N_{K/F}(K^\times)$ *for* $h_0 \in H_6(K)$ *of maximal index, where* N *is the reduced norm of* B.

Proof. First assume G is an inner form of G_0. We see $\mathrm{Ker}\,\rho_0 = \mu_3$, $Z(G_0) = \mu_6$, and

$$H^2(F, \mathrm{Ker}\,\rho_0) \to H^2\big(F, Z(G_0)\big)$$

is injective, and it is enough to determine $\delta^{-1}(H^2(F, \mathrm{Ker}\,\rho_0))$. In this case, we obtain (a).

Next assume G is an inner form of a quasisplit group ${}_aG_0$. Then ${}_aG_0$ is a special unitary group $\mathrm{SU}(h_0)$ for $h_0 \in H_6(K)$ for a quadratic extension K of F. We know $Z({}_aG_0) = R^{(1)}_{K/F}(\mu_6)$ and $\mathrm{Ker}\,{}_a\rho_0 = R^{(1)}_{K/F}(\mu_3)$. Since the exact sequence

$$1 \to R^{(1)}_{K/F}(\mu_3) \to R^{(1)}_{K/F}(\mu_6) \to R^{(1)}_{K/F}(\mu_2) \to 1$$

induced by the map $x \mapsto x^3$ of μ_6 to μ_2 splits, we see

$$H^2\big(F, R^{(1)}_{K/F}(\mu_3)\big) \to H^2\big(F, R^{(1)}_{K/F}(\mu_6)\big)$$

is injective, and we see it is enough to determine $\delta^{-1}(H^2(F, \mathrm{Ker}\,{}_a\rho_0))$. From the exact sequence

$$1 \to R^{(1)}_{K/F}(\mu_3) \to R_{K/F}(\mu_3) \to \mu_3 \to 1$$

we obtain

$$\begin{aligned} &H^1\big(F, R_{K/F}(\mu_3)\big) \\ &\quad \to H^1(F, \mu_3) \\ &\quad \to H^2\big(F, R^{(1)}_{K/F}(\mu_3)\big) \to H^2\big(F, R_{K/F}(\mu_3)\big) \to H^2(F, \mu_3); \end{aligned}$$

hence

$$\begin{aligned} 1 &\to F^\times/F^{\times 3}N_{K/F}(K^\times) \to H^2\big(F, R^{(1)}_{K/F}(\mu_3)\big) \\ &\to \mathrm{Ker}\big(H^2(K, \mu_3) \to H^2(F, \mu_3)\big) \to 1. \end{aligned}$$

We note an element of $\operatorname{Ker}(H^2(K, \mu_3) \to H^2(F, \mu_3))$ corresponds to a central simple algebra B over K with an involution of the second kind of index 1 or 3. For $c \in H^1(F, {}_a\overline{G}_0)$ satisfying $\delta(c) \in H^2(F, \operatorname{Ker}{}_a \rho_0)$, there exists a central simple algebra B over K of $\dim_K B = 6^2$, of index 1 or 3 with an involution of the second kind such that $G = \mathrm{SU}(h)$ for $h \in H_1(B)$. In the diagram

$$\begin{array}{ccccc}
H^2(F, R^{(1)}_{K/F}(\mu_3)) & \longrightarrow & H^2(F, R^{(1)}_{K/F}(\mu_6)) & \longrightarrow & H^2(F, R^{(1)}_{K/F}(\mu_2)) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_v H^2(F_v, R^{(1)}_{K/F}(\mu_3)) & \longrightarrow & \prod_v H^2(F_v, R^{(1)}_{K/F}(\mu_6)) & \longrightarrow & \prod_v H^2(F_v, R^{(1)}_{K/F}(\mu_2)),
\end{array}$$

all the vertical maps are injective. Hence we see the condition $\delta(c) \in H^2(F, R^{(1)}_{K/F}(\mu_3))$ for $c \in H^1(F, {}_a\overline{G}_0)$ can be checked locally.

Assume $G = \mathrm{SU}(h)$ for h as before. For $v \in \Sigma$ which splits in K, we have no condition. Let v be a place that does not split in K. Then we know

$$H^2\left(F_v, R^{(1)}_{K_v/F_v}(\mu_3)\right) \simeq F_v^\times / \left(F_v^{\times 3} N_{K_v/F_v}\left(K_v^\times\right)\right) = \mu_1$$

and we see $\delta(c)_v \in H^2(F_v, R^{(1)}_{K/F}(\mu_3))$ if and only if $\delta(c)_v$ is trivial. Since we have no division algebra with an involution of the second kind over local fields, we may assume $B_v = M_6(K_v)$ and $h \in H_6(K_v)$. From the exact sequence

$$1 \to \mathrm{SU}(h_0) \to \mathrm{U}(h_0) \to R^{(1)}_{K_v/F_v}(\mathbf{G}_m) \to 1,$$

we obtain

$$\begin{array}{ccccc}
H^1(F_v, \mathrm{SU}(h_0)) & \longrightarrow & H^1(F_v, \mathrm{U}(h_0)) & \longrightarrow & H^1(F_v, R^{(1)}_{K_v/F_v}(\mathbf{G}_m)) \\
\downarrow & & \downarrow & & \\
H^1(F_v, \overline{\mathrm{SU}(h_0)}) & \xrightarrow{\sim} & H^1(F_v, \overline{\mathrm{U}(h_0)}) & & \\
\downarrow{\scriptstyle\delta} & & \downarrow & & \\
H^2(F_v, Z(\mathrm{SU}(h_0))) & \longrightarrow & H^2(F_v, Z(\mathrm{U}(h_0))). & &
\end{array}$$

Under the identification

$$H^1\left(F_v, R^{(1)}_{K_v/F_v}(\mathbf{G}_m)\right) \simeq F_v^\times / N_{K_v/F_v}\left(K_v^\times\right),$$

the map $H^1(F_v, \mathrm{U}(h_0)) \to H^1(F_v, R^{(1)}_{K_v/F_v}(\mathbf{G}_m))$ is given by $a \mapsto N(h)/N(h_0)$, when a corresponds to $h \in H_6(K_v)$. Under the isomorphism $H^1(F_v,$

$\overline{\mathrm{SU}(h_0)}) \simeq H^1(F_v, \overline{\mathrm{U}(h_0)})$, c is the image of a class a in $H^1(F_v, \mathrm{U}(h_0))$ corresponding to h. From this, we obtain $\delta(c)_v$ is trivial if and only if $N(h)/N(h_0) \in N_{K_v/F_v}(K_v^\times)$. Since $N(h)/N(h_0) \in N_{K_v/F_v}(K_v^\times) \forall v \in \Sigma$ if and only if $N(h)/N(h_0) \in N_{K/F}(K^\times)$, we obtain our result. This completes the proof.

Remark 2.10. We see ${}_a\Theta_0^{\Gamma, \rho_0} = \mu_2$. Its nontrivial element gives an equivalence between the triplets for $\mathrm{SL}_2(D)$ and $\mathrm{SL}_2(D^0)$ in the case (a) and between the triplets for (B, h) and $(B^0, {}^\iota h)$ in the case (b), where ι is an anti-isomorphism of B to B^0.

THEOREM 2.11. *Let* (G, ρ, V) *be of type* I(15). *Let* F *be an algebraic number field or a non-archimedean local field.*

(1) *Assume n is odd or n is even* ≥ 6 *and m is odd. Then* (G, ρ, V) *is isomorphic to* $(\mathrm{Spin}(q) \times \mathrm{SL}_m, \bar{\rho}, M_{nm})$, *where* $q \in S_n(F)$ *with* $\mathrm{disc}(q) \neq 0$, *and*

$$\bar{\rho}((g_1, g_2))v = \rho_1(g_1)v\,{}^t g_2,$$

for the natural homomorphism ρ_1 *of* $\mathrm{Spin}(q)$ *to* $\mathrm{SO}(q)$.

(2) *Assume* $n \geq 6$ *and both n and m are even. Then* (G, ρ, V) *is equivalent to* (a) $(\mathrm{Spin}(q) \times \mathrm{SL}_m, \bar{\rho}, M_{nm})$, *where* $q \in S_n(F)$ *with* $\mathrm{disc}(q) \neq 0$ *and* $\bar{\rho}$ *as before, or to* (b) $(\mathrm{Spin}(h) \times \mathrm{SL}_{m/2}(D), \bar{\rho}, M_{n/2m/2}(D))$, *where D is a division quaternion algebra over F,* $h \in AH_{n/2}(D)$ *with* $\mathrm{disc}(h) \neq 0$, *and*

$$\bar{\rho}((g_1, g_2))v = \rho_1(g_1)v\tilde{g}_2,$$

with $\tilde{g}_1 = {}^t({}^\iota g_{ij})$ *for* $g = (g_{ij})$ *for the natural homomorphism* ρ_1 *of* $\mathrm{Spin}(h)$ *to* $\mathrm{SU}(h)$.

(3) *Assume* $n = 4$ *and* $m = 2$. *Then* (G, ρ, V) *is equivalent to* (a) $(\mathrm{SL}_1(D_1) \times \mathrm{SL}_1(D_2) \times \mathrm{SL}_1(D_3), \bar{\rho}, V(8))$, *where* D_1, D_2, *and* D_3 *are quaternion algebras over F such that* $c(D_1)c(D_2) = c(D_3)$ *and* $\bar{\rho}$ *is induced by the isomorphism* $D_1 \otimes D_2 \otimes D_3 \simeq M_8(F)$, *or to* (b) $(R_{K/F}(\mathrm{SL}_1(D_1)) \times \mathrm{SL}_1(D_2), \bar{\rho}, V(8))$, *where* D_1 *is a quaternion algebra over a quadratic extension K of F,* D_2 *is a quaternion algebra over F such that*

$$c(\mathrm{Cor}_{K/F}\, D_1) = c(D_2),$$

and $\bar{\rho}$ *is induced by the isomorphism* $\mathrm{Cor}_{K/F}(D_1) \otimes D_2 \simeq M_8(F)$, *or to* (c) $(R_{K/F}(\mathrm{SL}_1(D)), \bar{\rho}, V(8))$, *where D is a quaternion algebra over a cubic extension K of F such that* $c(\mathrm{Cor}_{K/F}(D))$ *is trivial and* $\bar{\rho}$ *is induced by the isomorphism* $\mathrm{Cor}_{K/F}(D) \simeq M_8(F)$. *Here* $\mathrm{Cor}_{K/F}(D)$ *denotes the corestriction of D for K/F* (*cf*. [R], *Theorem* 11).

Proof. When n is odd or n is even and m is odd, $\mathrm{Ker}_a\,\rho_0 = \mu_1$ or $\mu_2 \times \mu_1$ and the assertion can be proved easily.

Assume both m and n are even and $n \geq 6$. First assume 4 does not divide n and G is an inner form of G_0. Let $G_0 = H_1 \times H_2$, where $H_1 = \mathrm{Spin}(q_0)$ and $H_2 = \mathrm{SL}_m$. Then

$$Z(H_1) = \mu_4 = \langle z \rangle, \qquad Z(H_2) = \mu_m$$

and

$$\mathrm{Ker}\,\rho = \langle (z, z^2) \rangle \simeq \mu_4 .$$

We see $H^2(F, \mathrm{Ker}\,\rho) \to H^2(F, Z(G_0))$ is injective and

$$H^2(F, \mathrm{Ker}\,\rho) = \left\{ (a, a^2) | a \in H^2(F, \mu_4) \right\}.$$

Hence it is enough to determine $\delta^{-1}((a, a^2))$ for each $a \in H^2(F, \mu_4)$. From the exact sequence

$$1 \to Z(H_1)/Y_1 \to \mathrm{SO}(q_0) \to \overline{H}_1 \to 1,$$

we obtain the diagram

$$\begin{array}{ccccc} \mathrm{H}^1(\mathrm{F}, \mathrm{SO}(\mathrm{q}_0)) & \longrightarrow & \mathrm{H}^1(\mathrm{F}, \overline{\mathrm{H}}_1) & \xrightarrow{\eta} & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)/\mathrm{Y}_1) \\ \big\downarrow & & \big\downarrow & & \big\| \\ \mathrm{H}^2(\mathrm{F}, \mathrm{Y}_1) & \longrightarrow & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)) & \xrightarrow{\eta'} & \mathrm{H}^2(\mathrm{F}, \mathrm{Z}(\mathrm{H}_1)/\mathrm{Y}_1), \end{array} \tag{2.7}$$

where Y_1 is the kernel of the natural map $\mathrm{Spin}(q_0) \to \mathrm{SO}(q_0)$, and the lower horizontal sequence is induced by the exact sequence

$$1 \to Y_1 \to Z(H_1) \to Z(H_1)/Y_1 \to 1.$$

If $(c, c') \in H^1(F, \overline{H}_1) \times H^1(F, \overline{H}_2)$ belong to $\delta^{-1}((a, a^2))$, then $\eta(c) = a^2$ and a^2 determines a quaternion algebra D over F. If D splits, then the triplet is equivalent to (a) and if D does not split, it is equivalent to (b).

Assume G is an outer form of G_0 and let ${}_aG_0$ be the quasisplit group of which G is an inner form. Then ${}_aG_0 = \mathrm{Spin}(q)$, where q is an element in $S_n(F)$ of maximal index with $\mathrm{disc}(q) \neq 1$. Let K be the quadratic extension $F(\sqrt{\mathrm{disc}(q)})$ of F. Let ψ be the homomorphism of $R^{(1)}_{K/F}(\mu_4)$ to $R^{(1)}_{K/F}(\mu_2) = \mu_2$ induced by that of μ_4 to μ_2 $z \mapsto z^2$. Then we have $Z(H_1) = R^{(1)}_{K/F}(\mu_4)$ and

$$\mathrm{Ker}_a\,\rho_{0*} = \left\{ (a, \psi(a)) | a \in H^2(F, Z(H_1)) \right\}.$$

Then in the same way as before, we can prove our assertion.

Assume m is even and 4 divides n. Assume G is an inner form of G_0. Then we have

$$Z(H_1) = \mu_2 \times \mu_2 = \langle z \rangle \times \langle z' \rangle \supset Y_1 = \langle zz' \rangle$$

and

$$\operatorname{Ker} \rho_0 = \{(z, -1), (z', -1), (zz', 1), 1\}.$$

We have the same diagram as (2.7) and we see

$$H^2(F, \operatorname{Ker} \rho_0) = \{(c, c', cc') | c, c' \in H^2(F, \mu_2)\}.$$

We can check that the map η' in (2.7) is given by $\eta'(c, c') = cc'$ for $c, c' \in H^2(F, \mu_2)$. Assume G is an outer form of G_0 and assume G is an inner form of a quasisplit group ${}_aG_0$. Then ${}_aG_0$ is $\operatorname{Spin}(q)$ for $q \in S_n(F)$ of maximal index with $\operatorname{disc}(q) \neq 1$. Let K be as before. Then we know $Z(H_1) = R_{K/F}(\mu_2)$. Let N be the norm map of $R_{K/F}(\mu_2)$ to μ_2. Then we have

$$\operatorname{Ker}_a \rho_0 = \{(a, Na) | a \in R_{K/F}(\mu_2)\}.$$

In this case, η' is induced by N. We can proceed in the same way as before.

Assume $n = 4$ and $m = 2$. In this case, $G_0 \simeq \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and $\rho_0 = \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$. First consider the case of inner forms. Then we have

$$\operatorname{Ker} \rho_0 = \{(a, b, ab) | a, b \in \mu_2\}.$$

In this case we easily obtain (a). As for outer forms, we have to consider two cases: ${}_aG_0 = R_{K/F}\mathrm{SL}_2 \times \mathrm{SL}_2$ for a quadratic extension K of F and ${}_aG_0 = R_{K/F}\mathrm{SL}_2$ for a cubic extension K of F. In the first case, we have

$$\operatorname{Ker}_a \rho_0 = \{(a, Na) | a \in R_{K/F}(\mu_2)\},$$

where N is the norm map of $R_{K/F}(\mu_2)$ to μ_2, and in the latter case,

$$\operatorname{Ker}_a \rho_0 = \{a \in R_{K/F}(\mu_2) | Na = 1\}.$$

The map of $H^2(F, R_{K/F}(\mu_2)) = H^2(K, \mu_2)$ to $H^2(F, \mu_2)$ induced by the norm map corresponds to the corestriction for central simple algebras. From this, we easily obtain our result. This completes the proof.

Remark 2.12. If n or m is odd, $\Theta_0^{\Gamma, \rho} = \mu_1$, and if n is even and $n \geq 6$, then ${}_a\Theta_0^{\Gamma, \rho} = \mu_2$. In the cases of (1) and (2)(a), ${}_a\Theta_0^{\Gamma, \rho}$ acts on $H^1(F, {}_a\overline{G}_0)$ trivially, and in the case of (2)(b), it acts on classes of hermitian form trivially. In the case of (3), ${}_a\Theta_0^{\Gamma, \rho}$ equals the symmetric groups of degree 3, μ_2, the cyclic group of order 3, or μ_1 according to whether ${}_aG_0$ is of type (a), (b), or (c) for a cyclic extension K of F of degree 3, or (c) for a non-Galois cubic extension K of F. In the case of (a), $\Theta_0^{\Gamma,\rho}$ induces permutations among D_1, D_2, and D_3; in the other cases, the nontrivial elements do not affect G.

From the preceding results, we see $Z(G)$ is isomorphic to μ_n, $R_{K/F}(\mu_n)$ for a quadratic extension or a cubic extension K of F, $R^{(1)}_{K/F}(\mu_n)$ for a quadratic extension K of F, or a product of them. Let (G, ρ, V) be as before and set $\tilde{G} = G \times \mathbf{G}_m$ and $\tilde{\rho} = \rho \otimes 1$. Then $\operatorname{Ker} \tilde{\rho} \simeq Z(G)$ and we know the Hasse principle holds for $H^2(F, Z(G))$. Since the Hasse principle holds for $H^1(F, \tilde{G})$, in the same way as Theorem 6.22 of [P-R] (cf. the remark after the proof of Theorem 6.22), we can prove the following theorem.

THEOREM 2.13. *Let F be an algebraic number field and let (G, ρ, V) be the semisimple part of a reduced irreducible prehomogeneous vector space defined over F. Let $\tilde{G}$ and $\tilde{\rho}$ be as before. Then the Hasse map*

$$H^1\left(F, \tilde{G}/\operatorname{Ker} \tilde{\rho}\right) \to \prod_{v \in \Sigma} H^1\left(F_v, \tilde{G}/\operatorname{Ker} \tilde{\rho}\right)$$

is injective.

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